Linear systems – Final exam – Solutions

Final exam 2023–2024, Tuesday 18 June 2024, 15:00 – 17:00

Instructions

1. The use of books, lecture notes, or (your own) notes is not allowed.
2. All answers need to be accompanied with an explanation or calculation.

Problem 1

(6 + 8 + 14 + 6 = 34 points)

Consider the model of a tank in which two fluids are mixed, given as

$$\dot{h}(t) = \frac{q_C(t) + q_H(t) - c\sqrt{h(t)}}{A},$$
$$\dot{T}(t) = \frac{q_C(t)\big(T_C - T(t)\big) + q_H(t)\big(T_H - T(t)\big)}{Ah(t)}.$$

Here, h is the height of the fluid level of the tank and T is the temperature. The constants $c > 0$ and $A > 0$ represent the geometry of the tank, whereas T_C and T_H are the constant temperatures of the inflowing (cold and hot) fluids, with $0 < T_C < T_H$. Finally, q_C and q_H model the inflow and are regarded as control parameters. Thus, take

$$x(t) = \begin{bmatrix} h(t) \\ T(t) \end{bmatrix}, \qquad u(t) = \begin{bmatrix} q_C(t) \\ q_H(t) \end{bmatrix},$$

as the state and input, respectively.

(a) Assume that $q_C(t) > 0$, $q_H(t) > 0$ and $h(t) > 0$ for all t. Let $T(0)$ satisfy $T_C \leq T(0) \leq T_H$. Explain why $T_C \leq T(t) \leq T_H$ for all $t \geq 0$.

(b) Show that, for any desired equilibrium $h(t) = \bar{h}$, $T(t) = \bar{T}$ satisfying

$$\bar{h} > 0, \qquad T_C \leq \bar{T} \leq T_H,$$

there exists a unique constant input $q_C(t) = \bar{q}_C$, $q_H(t) = \bar{q}_H$ that achieves this equilibrium and satisfies $\bar{q}_C \geq 0$, $\bar{q}_H \geq 0$.

(c) Linearize the tank model around the equilibrium point $h(t) = \bar{h}$, $T(t) = \bar{T}$ and corresponding input $q_C(t) = \bar{q}_C$, $q_H(t) = \bar{q}_H$, as obtained in (b).

(d) Is the linearized system (internally) stable?

Answer Problem 1(a)

We will first show that $T(t) \leq T_H$. To this end, assume that $T(t) = T_H$. In this case,

$$\dot{T}(t) = \frac{q_C(t)\big(T_C - T(t)\big)}{Ah(t)} = \frac{q_C(t)\big(T_C - T_H\big)}{Ah(t)} < 0. \tag{1}$$

Hence, whenever $T(t) = T_H$, T needs to decrease. As solutions are continuous, this means that T can never grow above T_H and hence we have $T(t) \leq T_H$ for all $t \geq 0$. The bound $T_C \leq T(t)$ follows similarly.

Answer Problem 1(b)

For the desired equilibrium $h(t) = \bar{h}$, $T(t) = \bar{T}$, it holds that the time derivatives satisfy $\dot{h} = \dot{T} = 0$. Substituting this gives

$$0 = \bar{q}_c + \bar{q}_H - c\sqrt{\bar{h}}, \tag{2}$$

$$0 = \bar{q}_C\big(T_C - \bar{T}\big) + \bar{q}_H\big(T_H - \bar{T}\big). \tag{3}$$

In the above equations, the constant inputs $q_C(t) = \bar{q}_C$ and $q_H(t) = \bar{q}_H$ are used. Given the desired equilibrium $\bar{h}$, $\bar{T}$, the corresponding inputs can be obtained from the linear set of equations

$$\begin{bmatrix} 1 & 1 \\ T_C - \bar{T} & T_H - \bar{T} \end{bmatrix} \begin{bmatrix} \bar{q}_C \\ \bar{q}_H \end{bmatrix} = \begin{bmatrix} c\sqrt{\bar{h}} \\ 0 \end{bmatrix}, \tag{4}$$

as derived from rewriting (2) and (3). Since $T_C < T_H$, it holds that $T_C - \bar{T} < T_H - \bar{T}$, such that the matrix on the left-hand side of (4) is nonsingular. Hence, there exists a *unique* constant input $q_C(t) = \bar{q}_C$, $q_H(t) = \bar{q}_H$.

In fact, the solutions are readily computed as

$$\bar{q}_C = \frac{c\sqrt{\bar{h}}(T_H - \bar{T})}{T_H - T_C}, \tag{5}$$

$$\bar{q}_H = \frac{c\sqrt{\bar{h}}(\bar{T} - T_C)}{T_H - T_C}, \tag{6}$$

from which it can also be observed that $\bar{q}_C \geq 0$ and $\bar{q}_H \geq 0$ for the equilibrium satisfying $\bar{h} > 0$ and $T_C \leq \bar{T} \leq T_H$.

Answer Problem 1(c)

Define

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} h(t) \\ T(t) \end{bmatrix}, \qquad u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = \begin{bmatrix} q_C(t) \\ q_H(t) \end{bmatrix}, \tag{7}$$

and

$$f(x, u) = \begin{bmatrix} \frac{u_1 + u_2 - c\sqrt{x_1}}{A} \\ \frac{u_1(T_C - x_2) + u_2(T_H - x_2)}{Ax_1} \end{bmatrix}. \tag{8}$$

Then, the linearization of the dynamics (around the equilibrium $\bar{x}$ for constant input $\bar{u}$) is given by

$$\dot{\tilde{x}}(t) = \frac{\partial f}{\partial x}(\bar{x}, \bar{u})\tilde{x}(t) + \frac{\partial f}{\partial u}(\bar{x}, \bar{u})\tilde{u}(t), \tag{9}$$

where the perturbations are defined as

$$\tilde{x} = x - \bar{x}, \qquad \tilde{u} = u - \bar{u}. \tag{10}$$

Computation of the partial derivatives leads to

$$\frac{\partial f}{\partial x}(x, u) = \begin{bmatrix} -\frac{c}{2A\sqrt{x_1}} & 0 \\ -\frac{u_1(T_C - x_2) + u_2(T_H - x_2)}{Ax_1^2} & -\frac{u_1 + u_2}{Ax_1} \end{bmatrix}, \qquad \frac{\partial f}{\partial u}(x, u) = \begin{bmatrix} \frac{1}{A} & \frac{1}{A} \\ \frac{T_c - x_2}{Ax_1} & \frac{T_H - x_2}{Ax_1} \end{bmatrix}, \tag{11}$$

which can be evaluated at the equilibrium to obtain

$$\tilde{A} = \frac{\partial f}{\partial x}(\bar{x}, \bar{u}) = \begin{bmatrix} -\frac{c}{2A\sqrt{\bar{h}}} & 0 \\ -\frac{\bar{q}_C(T_C - \bar{T}) + \bar{q}_H(T_H - \bar{T})}{A\bar{h}^2} & -\frac{\bar{q}_C + \bar{q}_H}{A\bar{h}} \end{bmatrix} = \begin{bmatrix} -\frac{c}{2A\sqrt{\bar{h}}} & 0 \\ 0 & -\frac{c}{A\sqrt{\bar{h}}} \end{bmatrix}, \tag{12}$$

$$\tilde{B} = \frac{\partial f}{\partial u}(\bar{x}, \bar{u}) = \begin{bmatrix} \frac{1}{A} & \frac{1}{A} \\ \frac{T_c - \bar{T}}{A\bar{h}} & \frac{T_H - \bar{T}}{A\bar{h}} \end{bmatrix}. \tag{13}$$

Then, the linearized dynamics is given as

$$\dot{\tilde{x}}(t) = \tilde{A}\tilde{x}(t) + \tilde{B}\tilde{u}(t), \tag{14}$$

with the matrices $\tilde{A}$ and $\tilde{B}$ as above. Note that these matrices are constant (as the equilibrium point is chosen).

Answer Problem 1(d)

The stability is determined by the eigenvalues of $\tilde{A}$ in (12), which equal its diagonal elements due to the diagonal structure of $\tilde{A}$. Since $c > 0$ and $\bar{h} > 0$, it is clear that the eigenvalues are real-valued and (strictly) negative, such that the system is (internally) stable.

Problem 2 (4 + 10 + 10 = 24 points)

Consider the linear system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad \text{with} \quad A = \begin{bmatrix} 8 & 5 \\ -10 & -6 \end{bmatrix}, B = \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

and where $x(t) \in \mathbb{R}^2$ and $u(t) \in \mathbb{R}$.

(a) Verify that the system is controllable.

(b) Find a nonsingular matrix T and real numbers α_1, α_2 such that

$$TAT^{-1} = \begin{bmatrix} 0 & 1 \\ \alpha_1 & \alpha_2 \end{bmatrix}, \qquad TB = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

It is sufficient to give T^{-1}, but follow a systematic procedure.

(c) Use the matrix T from (b) to design a state feedback controller $u(t) = Fx(t)$ such that the resulting closed-loop system satisfies $\sigma(A + BF) = \{-2, -3\}$.

Answer Problem 2(a)

To verify controllability, compute

$$\begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ -1 & -4 \end{bmatrix} \tag{15}$$

and note that

$$\text{rank} \begin{bmatrix} B & AB \end{bmatrix} = \text{rank} \begin{bmatrix} 1 & 3 \\ -1 & -4 \end{bmatrix} = 2, \tag{16}$$

which equals the state-space dimension. Hence, the system is controllable.

Answer Problem 2(b)

First, compute the characteristic polynomial as

$$\Delta_A(s) = \det(sI - A) = \begin{vmatrix} s-8 & -5 \\ 10 & s+6 \end{vmatrix} = (s-8)(s+6) + 50 = s^2 - 2s + 2, \tag{17}$$

which can be written as

$$\Delta_A(s) = s^2 + a_1 s + a_0, \tag{18}$$

with

$$a_1 = -2, \quad a_0 = 2. \tag{19}$$

As the pair (A, B) is controllable, there exists a nonsingular matrix T such that

$$TAT^{-1} = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix}, \qquad TB = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \tag{20}$$

which is the controllability canonical form. By comparing this to the matrices in the problem statement, we can choose α_1 and α_2 such that

$$\alpha_1 = -a_0 = -2, \qquad \alpha_2 = -a_1 = 2. \tag{21}$$

To find the matrix T that achieves the transformation, define

$$q_2 = B = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \tag{22}$$

and

$$q_1 = AB + a_1 B = \begin{bmatrix} 3 \\ -4 \end{bmatrix} + (-2) \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}. \tag{23}$$

Now, noting that q_1 and q_2 are linearly independent, we can define T by specifying its inverse as

$$T^{-1} = \begin{bmatrix} q_1 & q_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}. \tag{24}$$

For completeness, note that

$$T = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix}, \tag{25}$$

after which it is readily verified that

$$TAT^{-1} = \begin{bmatrix} 0 & 1 \\ -2 & 2 \end{bmatrix}, \qquad TB = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{26}$$

Answer Problem 2(c)

As a first step, we define a polynomial p with roots at the desired eigenvalues for $A + BF$. This leads to

$$p(s) = (s+2)(s+3) = s^2 + 5s + 6, \tag{27}$$

which can be written as

$$p(s) = s^2 + p_1 s + p_0, \tag{28}$$

with

$$p_1 = 5, \quad p_0 = 6. \tag{29}$$

Our objective is to find a matrix F such that

$$\Delta_{A+BF}(s) = p(s). \tag{30}$$

To achieve this, note that

$$\Delta_{A+BF}(s) = \Delta_{T(A+BF)T^{-1}}(s) = \Delta_{TAT^{-1}+TBFT^{-1}}(s) \tag{31}$$

for any nonsingular matrix T. Using the matrix T from problem (b), this gives

$$TAT^{-1} = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix}, \qquad TB = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{32}$$

Next, denote

$$FT^{-1} = \begin{bmatrix} f_0 & f_1 \end{bmatrix}, \tag{33}$$

such that

$$TAT^{-1} + TBFT^{-1} = \begin{bmatrix} 0 & 1 \\ f_0 - a_0 & f_1 - a_1 \end{bmatrix}. \tag{34}$$

As this matrix is in companion form, we can easily obtain its characteristic polynomial as

$$\Delta_{TAT^{-1}+TBFT^{-1}}(s) = s^2 + (a_1 - f_1)s + (a_0 - f_0). \tag{35}$$

Hence, after recalling (31), we see that the objective (30) is achieved if and only if

$$a_1 - f_1 = p_1, \qquad a_0 - f_0 = p_0. \tag{36}$$

Solving these equations leads to

$$f_1 = a_1 - p_1 = -2 - 5 = -7, \qquad f_0 = a_0 - p_0 = 2 - 6 = -4. \tag{37}$$

Then, to obtain the feedback matrix F, we need to solve the linear equation

$$FT^{-1} = \begin{bmatrix} f_0 & f_1 \end{bmatrix}, \tag{38}$$

which reads

$$F \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} = \begin{bmatrix} -4 & -7 \end{bmatrix}, \tag{39}$$

and leads to

$$F = \begin{bmatrix} -10 & -3 \end{bmatrix}. \tag{40}$$

Finally, we can verify that

$$A + BF = \begin{bmatrix} -2 & 2 \\ 0 & -3 \end{bmatrix}, \tag{41}$$

such that $\sigma(A + BF) = \{-2, -3\}$ as desired.

Problem 3 (8 + 8 + 10 + 6 = 32 points)

Consider the linear system

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{42}$$

with state $x(t) \in \mathbb{R}^n$ and input $u(t) \in \mathbb{R}^m$. Denote by $x(t; x_0, u)$ the state trajectory for given initial condition $x(0) = x_0$ and input function $u : [0, \infty) \to \mathbb{R}^m$, i.e.,

$$x(t; x_0, u) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)\,\mathrm{d}\tau.$$

Define the stabilizable subspace

$$\mathcal{S} = \left\{x_0 \in \mathbb{R}^n \;\middle|\; \text{there exists } u : [0, \infty) \to \mathbb{R}^m \text{ such that } \lim_{t\to\infty} x(t; x_0, u) = 0\right\}.$$

(a) Show that $\mathcal{S}$ is a subspace of $\mathbb{R}^n$.

(b) Explain that $\mathcal{S} = \mathbb{R}^n$ if the system (42) is stabilizable.

(c) Show that $\mathcal{W} \subset \mathcal{S}$, where $\mathcal{W}$ is the reachable subspace

$$\mathcal{W} = \left\{x_T \in \mathbb{R}^n \;\middle|\; \text{there exists } T > 0 \text{ and } u : [0, T] \to \mathbb{R}^m \text{ such that } x(T; 0, u) = x_T\right\}.$$

Hint. Recall that $\mathcal{W}$ is A-invariant. Note that the notation $\mathcal{W} \subset \mathcal{S}$ allows for $\mathcal{W} = \mathcal{S}$ and that we do not assume stabilizability here.

(d) Give an example of a system in which the inclusion $\mathcal{W} \subset \mathcal{S}$ is *strict*, i.e., we have $\mathcal{W} \subset \mathcal{S}$ and $\mathcal{W} \neq \mathcal{S}$.

Answer Problem 3(a)

To show that $\mathcal{S}$ is a subspace, we need to show that it contains 0 and is closed under addition and scalar multiplication.

First, we take $x_0 = 0$ and let $u(t) = 0$ for all $t \geq 0$. Then, it is clear that $x(t; 0, 0) = 0$ for all $t \geq 0$, such that we have

$$0 \in \mathcal{S} \tag{43}$$

by the definition of $\mathcal{S}$.

To show closure under addition, let $x_0,\, x_0' \in \mathcal{S}$. By definition, this means that there exists input functions $u,\, u' : [0, \infty) \to \mathbb{R}^m$ such that

$$\lim_{t\to\infty} x(t; x_0, u) = 0 \qquad \text{and} \qquad \lim_{t\to\infty} x(t; x_0', u') = 0. \tag{44}$$

Then, after recalling that

$$x(t; x_0, u) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)\,\mathrm{d}\tau, \tag{45}$$

$$x(t; x_0', u') = e^{At}x_0' + \int_0^t e^{A(t-\tau)}Bu'(\tau)\,\mathrm{d}\tau, \tag{46}$$

it is clear that

$$\begin{aligned} x(t; x_0 + x_0', u + u') &= e^{At}(x_0 + x_0') + \int_0^t e^{A(t-\tau)}\big(u(\tau) + u'(\tau)\big)\,\mathrm{d}\tau, \\ &= x(t; x_0, u) + x(t; x_0', u'). \end{aligned} \tag{47}$$

As a result

$$\begin{aligned}\lim_{t\to\infty} x(t;x_0+x_0',u+u') &= \lim_{t\to\infty} x(t;x_0,u)+x(t;x_0',u'),\\ &= \lim_{t\to\infty} x(t;x_0,u)+\lim_{t\to\infty} x(t;x_0',u')=0,\end{aligned} \tag{48}$$

such that

$$x_0+x_0' \in \mathcal{S}. \tag{49}$$

Finally, to show closure under scalar multiplication, let $x_0 \in \mathcal{S}$ and take $c \in \mathbb{R}$. As before, the definition of $\mathcal{S}$ guarantees that there exists $u:[0,\infty)\to\mathbb{R}^m$ such that

$$\lim_{t\to\infty} x(t;x_0,u)=0. \tag{50}$$

After recalling (45), it is clear that

$$x(t;cx_0,cu)=cx(t;x_0,u), \tag{51}$$

and we have

$$\lim_{t\to\infty} x(t;cx_0,cu)=\lim_{t\to\infty} cx(t;x_0,u)=c\lim_{t\to\infty} x(t;x_0,u)=0. \tag{52}$$

Thus, we have shown that

$$cx_0 \in \mathcal{S}. \tag{53}$$

Answer Problem 3(b)

Assume that (42) is stabilizable. Then, there exists a matrix F such that $\sigma(A+BF)\subset\mathbb{C}_-$. As a result, the controlled system

$$\dot{z}(t)=(A+BF)z(t) \tag{54}$$

satisfies

$$\lim_{t\to\infty} z(t;x_0)=\lim_{t\to\infty} e^{(A+BF)t}x_0=0 \tag{55}$$

for all $x_0\in\mathbb{R}^n$. After defining $u(t)=Fz(t)$, we can view the solution $z(t;x_0)$ for a given $x_0\in\mathbb{R}^n$ as the solution $x(t;x_0,u)$ to (42) for the choice of input

$$u(t)=Fe^{(A+BF)t}x_0. \tag{56}$$

In other words, we have

$$z(t;x_0)=x\Big(t;x_0,Fe^{(A+BF)t}x_0\Big). \tag{57}$$

Then, by (55), we have that

$$\lim_{t\to\infty} x\Big(t;x_0,Fe^{(A+BF)t}x_0\Big)=0 \tag{58}$$

for any $x_0\in\mathbb{R}^n$, which shows that $\mathcal{S}=\mathbb{R}^n$.

Just for fun, we explicitly show (57). To this end, choose any $x_0\in\mathbb{R}^n$ and introduce the short-hand notation

$$x(t)=x\Big(t;x_0,Fe^{(A+BF)t}x_0\Big). \tag{59}$$

Next, define

$$\xi(t) = x(t) - z(t; x_0). \tag{60}$$

Note that

$$\xi(0) = x(0) - z(0; x_0) = x_0 - x_0 = 0. \tag{61}$$

Next, we emphasize that

$$\tfrac{\mathrm{d}}{\mathrm{d}t} x(t) = Ax(t) + BFe^{(A+BF)t}x_0, \tag{62}$$

for all $t \geq 0$, as $x(t)$ is nothing more than the solution to (42) for input (56) (and initial condition x_0). Using that $z(t; x_0)$ is the solution to (54) and therefore satisfies

$$\tfrac{\mathrm{d}}{\mathrm{d}t} z(t; x_0) = (A + BF)z(t; x_0) \tag{63}$$

for all $t \geq 0$, we obtain

$$\dot{\xi}(t) = Ax(t) + BFe^{(A+BF)t}x_0 - (A + BF)z(t; x_0) = Ax(t) - Az(t; x_0) = A\xi(t). \tag{64}$$

Thus, $\xi(t)$ satisfies the differential equation (64) with initial condition (61). This has the unique solution

$$\xi(t) = 0 \tag{65}$$

for all $t \geq 0$, which proves (57).

Answer Problem 3(c)

To show that $\mathcal{W} \subset \mathcal{S}$, take $x_0 \in \mathcal{W}$. Fix some $T > 0$. Then, as $\mathcal{W}$ is A-invariant, we also have

$$e^{AT}x_0 \in \mathcal{W}, \tag{66}$$

and, moreover,

$$-e^{AT}x_0 \in \mathcal{W}. \tag{67}$$

Recalling that $\mathcal{W}$ characterizes the subspace of states that can be reached from zero, we have that there exists an input function $u_T : [0, T] \to \mathbb{R}^m$ such that

$$-e^{AT}x_0 = \int_0^T e^{A(T-\tau)}Bu_T(\tau)\,\mathrm{d}\tau. \tag{68}$$

Rearranging terms gives

$$0 = e^{AT}x_0 + \int_0^T e^{A(T-\tau)}Bu_T(\tau)\,\mathrm{d}\tau. \tag{69}$$

The right-hand side can be recognized as the solution to the linear system for initial condition x_0 and input function u_T, i.e., we have

$$x(t; x_0, u_T) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu_T(\tau)\,\mathrm{d}\tau \tag{70}$$

for all $t \in [0, T]$, such that (69) gives

$$x(T; x_0, u_T) = 0. \tag{71}$$

Now, define the input function

$$u(t) = \begin{cases} u_T(t), & t \in [0, T], \\ 0, & t > T. \end{cases} \tag{72}$$

Then, it is clear that

$$x(t; x_0, u) = \begin{cases} x(t; x_0, u_T), & t \in [0, T], \\ 0, & t > T, \end{cases} \tag{73}$$

where the solution for $t > T$ follows from noting (71). In other words, the input u is such that

$$\lim_{t \to \infty} x(t; x_0, u) = 0, \tag{74}$$

which shows that $x_0 \in \mathcal{S}$. As x_0 was chosen arbitrarily in $\mathcal{W}$, we have $\mathcal{W} \subset \mathcal{S}$.

Answer Problem 3(d)

An example is

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad \text{with} \qquad A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \, B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{75}$$

Namely, it is clear that

$$\mathcal{W} = \operatorname{im} \begin{bmatrix} B & AB \end{bmatrix} = \operatorname{im} \begin{bmatrix} -1 & -1 \\ 0 & 0 \end{bmatrix} = \operatorname{span} \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}. \tag{76}$$

On the other hand, as $\sigma(A) \subset \mathbb{C}_-$, we have

$$\lim_{t \to \infty} x(t; x_0, 0) = \lim_{t \to \infty} e^{At} x_0 = 0, \tag{77}$$

for any $x_0 \in \mathbb{R}^n$. Note that we have chosen $u(t) = 0$ for all $t \geq 0$ here. Thus, we conclude that

$$\mathcal{S} = \mathbb{R}^n, \tag{78}$$

and we clearly have

$$\mathcal{W} \subset \mathcal{S} \qquad \text{and} \qquad \mathcal{W} \neq \mathcal{S} \tag{79}$$

as desired.

(10 points free)